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CIRCULAR REPRESENTATION PROBLEM ON HYPERGRAPHS*

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A hypergraph $J=(X, E)$ is said to be circular representable, if its vertices can be placed on a circle, in such way that every edge of H induces an interval. This concept is a translation into the vocabulary of hypergraphs of the circular one's property for the (0, 1) matrices [6] studied by Tucker [9, 10]. We give here a characterization of the hypergraphs which are circular representable. We study when the associated representation is unique, and we characterize the possible transformations of a representation into another, a kind of problem which has already been treated from the algorithmic point of view by Booth and Lueker [1] or Duchet [2] in the case of the interval representable hypergraphs.

Finally, we establish a connection between circular graphs and circular representable hypergraphs of the type of the Fulkerson–Gross connection between interval graphs and matrices having the consecutive one's property [5], in some special cases.

Preliminaries and notation

A hypergraph $H=(X, E)$ is a finite set X (vertex set), given with a collection E of subsets of X (edges).

If $A \subset X$ is a subset of X, we write H_A the subhypergraph of H induced by A.

A graph $G=(X, E)$ is a hypergraph whose edges are all with cardinality 2.

An edge of a graph 'between' 2 vertices x and y is usually written $[x, y]$.

A clique of a graph $G=(X, E)$ is a subset $A \subset X$ such that every pair x, y in A form an edge of G.

A triangle of a graph G is a clique with at most 3 vertices. A triangle is not degenerate if it contains exactly 3 vertices.

1. Circular representation of a hypergraph

A finite hypergraph $H=(X, E)$ will be said to be *circular-representable*, if there exists a function p from X to a set of $n=|X|$ points $1, 2, \ldots, n$ placed in this

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order on an oriented circle C, such that

(i) p is one-to-one;

(ii) $\forall A \in E$, $p(A)$ is constituted by consecutive points on the circle C. (We shall say: $p(A)$ is an interval.)

Such a function p will be called a circular-representation of H, and the set of all these functions p will be denoted by $R(H)$.

We shall say that a hypergraph $H=(X, E)$ has the *anti-Helly property* if $\forall E_1, E_2, E_3 \in E$ such that $\bigcap_{i=1}^{3} E_i \neq \emptyset$, there exists $i \in 1, 2, 3$ such that

$$E_i \subset E_j \cup E_k, \quad j, k \in 1, 2, 3, \ i \neq j, k.$$

If X is a finite set, we can compare two hypergraphs $H=(X, E)$ and $H'=(X, E')$, saying that H is smaller than H' if $E \subset E'$.

If $H=(X, E)$ is a finite hypergraph, we shall denote by $\bar{H}=(X, \bar{E})$ the smallest hypergraph (in the above sense) $H'=(X, E')$ such that

$$\begin{aligned}
&E \subset E';\\
&(A \subset X, |A|=0, 1, n-1 \text{ or } n \text{ with } n=|X|) \Rightarrow A \in E';\\
&E_1, E_2 \in E',\ E_1 \cap E_2 \neq \emptyset \Rightarrow E_1 \cup E_2 \in E';\\
&E_1, E_2 \in E',\ E_1 \cup E_2 \neq X \Rightarrow E_1 \cap E_2 \in E';\\
&E_1, E_2 \in E',\ E_1 \not\subset E_2 \Rightarrow E_2 - E_1 \in E'.
\end{aligned}$$

Then we get the following result:

Theorem 1.1. *The hypergraph $H=(X, E)$ is circular-representable if and only if we have both the properties*:

(1) *The hypergraph $\bar{H}$ satisfies the anti-Helly property.*

(2) *There exists a vertex x_0 in H, such that the partial hypergraph of $\bar{H}$ constituted by the edges which do not contain x_0 is interval-representable.*

Recall that a finite hypergraph $H=(X, E)$ is interval-representable if E can be considered as a family of intervals for X linearly ordered.

The problem of characterizing the hypergraphs which are interval-representable has already been treated, by Duchet [3, 4], Trotter and Moore [7], Tucker [9] and many others. Let us give for example the following result.

Theorem (Duchet [3, 4]). *A hypergraph $H=(X, E)$ is interval-representable if and only if we have: The hypergraph of the chains of H,*

$$C(H)=\left(X, \bigcup_{i=1}^{p} E_i \mid E_1, \ldots, E_p \text{ is a chain of } H\right)$$

doesn't contain K_3 as a partial subhypergraph.

Proof of Theorem 1.1. The 'only if' part of the claim is obvious.

'If' part. We proceed by induction on $n=|X|$, and we assume that $H=\bar{H}$.

(A) Let us suppose that there exists a subset $a \subset X$ such that

$$x_0 \notin A, \qquad 2 \leqslant |A| < n;$$

$$\forall B \in E, \quad A \subset B \text{ or } B \subset A \text{ or } B \cap A = \emptyset \text{ or } |B| = n-1.$$

Then we consider the hypergraph $H(A)$ obtained by confusing all the vertices of A into one unique vertex a: $T \subset (X-A) \cup \{a\}$ is an edge of $H(A) \Leftrightarrow$ There exists $T' \in E$ such that if $T' \cap A = \emptyset$, then $T' = T$ and if $T' \cap A \neq \emptyset$, then $(T'-A) \cup \{a\} = T$.

We obviously have: $\overline{H(A)} = H(A)$. It is also clear that the partial hypergraph of $H(A)$ formed by the edges which do not contain x_0 is interval-representable.

Let us prove that $H(A)$ has the anti-Helly property: We consider 3 edges T_1, T_2, T_3 of $H(A)$ such that

$$T_1 \cap T_2 \cap T_3 \neq \emptyset.$$

They come from 3 edges T_1', T_2', T_3' of H. It is clear that $T_i' \subset T_j' \cup T_k'$ implies $T_i \subset T_j \cup T_k$. So we assume $T_1' \cap T_2' \cap_3' = \emptyset$ (nontrivial case). That means $T_1 \cap T_2 \cap T_3 = \{a\}$ and, because of the definition of A, that one of the T_i is equal to $\{a\}$ or that $|T_1'| = |T_2'| = |T_3'| = n-1$. In both cases, we get the result.

Then by induction, $H(A)$ is circular-representable, and we may represent $H(A)$ on a circle. Besides, the edges of H which are contained in A induce a subhypergraph partial of H on A which is interval-representable. We replace the vertex a of $H(A)$ by the vertices of A ordered according to such an interval-representation, and we clearly get the circular representation for the hypergraph H. See Fig. 1.

Henceforth, we suppose that there is not $A \subset X$ that satisfies the (A)-hypothesis.

(B) Let us now suppose that there exists in H an edge E_0, with cardinality 2, which doesn't contain x_0; we denote this edge by (a, b). According to (A), there exists an edge of H, containing for instance a and not containing b, and with cardinality different from 1, $n-1$, n. Let us call I_a the set of such edges of H. Then I_a is a total order for the inclusion (by anti-Helly property). Let us call T_0 the minimal element of I_a ($|T_0| \geqslant 2$), and let us consider the subhypergraph H_{X-a}.

It is clear that H_{X-a} satisfies the hypothesis (2) of Theorem 1.1, and that

$$H_{X-a} = \overline{H_{X-a}}.$$

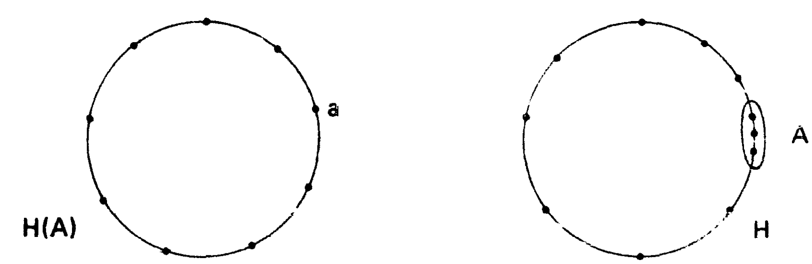

Fig. 1.

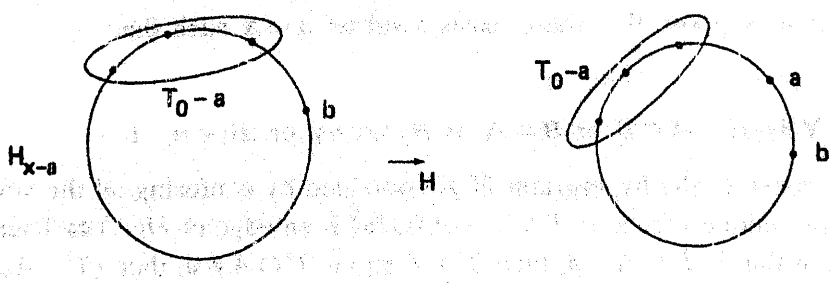

Fig. 2.

Let us prove that H_{X-a} satisfies the anti-Helly property: If T_1, T_2, T_3 are 3 edges in H_{X-a} such that

$$T_1 \cap T_2 \cap T_3 \neq \emptyset,$$

then there exist 3 edges T_1', T_2', T_3' of H such that

$$\forall i \in \{1, 2, 3\}, \quad T_i' - \{a\} = T_i; \qquad T_1' \cap T_2' \cap T_3' \neq \emptyset.$$

The following implication is true:

$$T_i' \subset T_j' \cup T_k' \Rightarrow T_i \subset T_j \cup T_k.$$

Thus, H_{X-a} is circular (induction), and may be represented on a circle. It is clear that $T_0 - a$ and $(T_0 - a) \cup \{b\}$ are edges in H_{X-a}, so in our representation of H_{X-a}, we have that $T_0 - a$ and b are consecutive. We place a between $T_0 - a$ and b, as shown in Fig. 2. An edge of H containing a, contains either b, or T_0, or is reduced to a (because $\leq_a$ is a total order), and therefore is represented above as an interval. An edge F of H which doesn't contain a can't be 'broken' by a, in the representation above. That would mean that

$$F \cap (T_0 - a) \neq \emptyset \quad \text{and} \quad b \in F.$$

Then the 3 edges of H, F, T_0, and $\{a, b\}$ are such that (see Fig. 3)

$$F \cap T_0 \neq \emptyset, \qquad F \cap \{a, b\} \neq \emptyset, \qquad T_0 \cap \{a, b\} \neq \emptyset,$$
$$F \cap T_0 \cap \{a, b\} = \emptyset.$$

Because of the hypothesis (2), that means that $F = X - a$ and the problem is then solved.

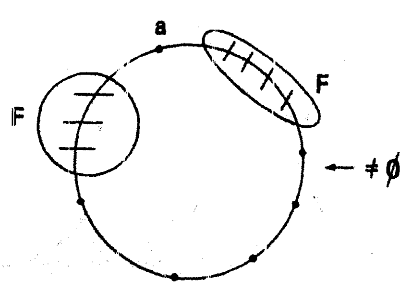

Fig. 3.

(C) The only thing we have to do is to prove that there exists in H an edge with cardinality 2 (supposing of course that (A) can't be applied). Let us consider an edge E_0, with cardinality $\geqslant 2$, which doesn't contain x_0 and which is minimal for the inclusion order with these properties. Of course we suppose that $|E_0| \geqslant 3$, and because of (A), we can say that there exists an edge B of H such that

$$B \cap E_0 \neq \emptyset, \qquad |B| \neq n-1,$$
$$B \not\subset E_0, \qquad E_0 \not\subset B.$$

$E_0 - B$ is an edge of H, and must then be with cardinality 1. Then $E_0 \cap B$ is with cardinality $\geqslant 2$, and will contradict the minimality of E_0, except if $E_0 \cup B = X$. That means that $|B|$ is in fact equal to $n-1$ and we get a contradiction and the result. However, E_0 may not exist. That will mean that all nontrivial edges of H (with cardinality other than 1, $n-1$, n) contain x_0. In that case, the last step of our proof will be to apply the following lemma. □

Lemma. *If $H = (X, E)$ is an hypergraph such that*
(i) $\bigcap_{A \in E} A \neq \emptyset$;
(ii) $\forall A, B \in E$, *we have*: $A \subset B$, $B \subset A$ *or* $|A - B| = |B - A| = 1$;
(iii) $\forall A$, $B \in E$, $A \cup B \in E$,
then H is circular-representable.

It is clear that if every nontrivial edge of H contains x_0, then H satisfies (i), (ii), (iii) of the lemma.

Proof of the lemma. We proceed by induction on $|X|$, and that leads us to suppose that: $|\bigcap_{A \in E} A| = 1$.

If we denote by x_0 the vertex at the intersection of the edges of H, it becomes easy, using (ii), to prove that there exist x and y in X such that

$$A_1 = (x, x_0) \in E, \qquad A_2 = (y, x_0) \in E;$$

Every edge in H which is different from x_0, A_1, A_2, contains x, y, x_0. Then we fuse x, y, x_0 into one vertex, and we conclude by induction. □

2. Unicity of representation

We now consider a finite hypergraph $H = (X, E)$ supposed to be circular-representable. Our purpose here is to find out when the circular representation of H is unique (modulo the obvious rotations and symmetries on the set $(1, 2, \ldots, n)$ in this order on the circle, $n = |X|$), and to characterize the transformations which permit passing from a representation to another. We shall apply these results, in order to obtain some corollaries about the interval-representable hypergraphs; similar results may be found in [1] or [2]. Duchet [2] gives a method to compute the number of interval representations of an interval-representable hypergraph; Booth and Lueker [1] give an efficient algorithm which permits practically to obtain all the interval representations of an interval-representable hypergraph and using Tucker's transformation [9], to obtain the circular representations of a circular-representable hypergraph.)

We denote by $R(H)$ the set of the circular representation of H. If $p, p' \in R(H), pp'^{-1}$ is a permutation of the set $(1, 2, \ldots, n)$. In general, the set of these permutations doesn't form a subgroup of S_n, but contains the subgroup D_n generated by

(i) the 'inversion' v: $v(1)=n$; $v(2)=n-1, \ldots, v(n)=1$;
(ii) the 'inversion' v': $v'(1)=1$; $v'(2)=n, \ldots, v'(n)=2$;
(iii) all the circular permutations of S_n.

Definition. The circular representation of H will be said to be *unique*, if the set of the permutations pp'^{-1} $(p, p' \in R(H))$ is exactly the subgroup D_n.

We shall say now that a subset $A \subset X$ is *vague* if it is possible to write:

$$\forall B \in E, \quad B \subset A \text{ or } A \subset B \text{ or } B \cap A = \emptyset \text{ or } B \cup A = X.$$

Then we get the following result:

Theorem 2.1. *The circular representation of the hypergraph $H=(X, E)$ is unique if and only if the only vague subsets in H are the subsets of X with* 0, 1, $n-1$ *or* n *elements.*

Proof. 'Only if' part. Let us consider that p is a representation of H, and that $A \subset X$ is a vague subset in H, with $|A| \neq 0$, 1, $n-1$, n, and minimal for the inclusion with these properties. If $p(A)$ is an interval, we pose $p(A)=i_1, i_2, \ldots, i_k$ in this order on the circle, and we construct another representation p' of H as follows:

$$x \notin A \Rightarrow p(x)=p'(x);$$
$$x \in A \Rightarrow p'(x)=i_{k+1-u(x)},$$

where $u(x)$ is defined by $p(x)=i_{u(x)}$.

p' is clearly an element of $R(H)$ and pp'^{-1} is not in D_n.

If $p(A)$ is not an interval, but is the union of intervals $p(A_j)$, then every A_j is vague, and must be supposed with cardinality 1. In that case, we resolve the problem considering $X-A$, which is also a vague subset in H. The only difficulty arises when $n=|X|$ is even and when $p(A)$ is for instance the set $(1, 3, 5, \ldots, n-1)$. But in that case, it becomes easy to see that every edge in H is with cardinality 1, $n-1$ or n, and to conclude.

'If' part. Let us call $\bar{H}$ the smallest hypergraph $H'=(X, E')$ (in the sense defined in Section 1) such that

$$E \subset E',$$
Every subset of cardinality $(n-1)$ in X is in E,
$$E_1, E_2 \in E', E_1 \cap E_2 \neq \emptyset \Rightarrow E_1 \cup E_2 \in E',$$
$$E_1, E_2 \in E', E_1 \cup E_2 \neq X \Rightarrow E_1 \cap E_2 \in E',$$
$$E_1, E_2 \in E', E_1 \not\subset E_2 \Rightarrow E_2 - E_1 \in E',$$
$$E_1 \in E' \Rightarrow E_1^c \in E',$$
$$X \in E'.$$

(In comparison with Theorem 1.1, we add here the property: $E_1 \in E' \Rightarrow E_1^c \in E$!)

Of course $\bar{H}$ exists. It is almost obvious that every circular representation for $\bar{H}$ is a representation for H and conversely. Thus we have $R(H) = R(\bar{H})$.

Lemma. *If $A \subset X$ is a vague subset for H, it is also a vague subset for $\bar{H}$ and conversely.*

Proof left to the reader. (It is sufficient to use the fact that we can pass from H to $\bar{H}$ by successive additions of one edge.)

Returning now to our main claim, the 'if' part of Theorem 2.1, we see that we may assume that $H = \bar{H}$. We also suppose that H has no vague subset other than the trivial subsets. Then we consider $x_0 \in X$, and an edge T_0 of H, which contains x_0, with $|T_0| \geq 2$, and which is minimal for the inclusion with these properties. If $|T_0| = n-1$, we consider $x_1 \in X - T_0$; If there exists a nontrivial edge B which contains x_1 and doesn't contain x_0, then we get a contradiction on the minimality of T_0 by considering $T_0 - B$. Thus we must suppose that every nontrivial edge containing x_1 contains x_0. But in that case we see that $\{x_0, x_1\}$ is a vague subset of H, which brings a contradiction. Thus we have $2 \leq |T_0| \leq n-2$. Now we can say:

$$T_0 \text{ is not vague} \Rightarrow \exists B \in E,\ B \cap T_0 \neq B, T_0, \emptyset,\ B \cup T_0 \neq X,$$
$$\Rightarrow T_0 - B \text{ and } T_0 \cap B \in E,$$

and because one of these edges contains x_0, it must be reduced to x_0 (definition of T_0); it implies $T_0' = T_0 - x_0 \in E$.

In a representat. .ı of H, we have the following situation (Fig. 4).

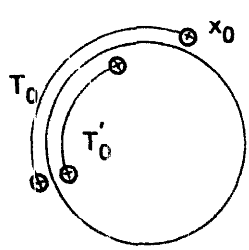

Fig. 4.

If $|T_0| \geq 3$, then $|T_0'| \geq 2$. Then T_0' is not vague, so there exists $B \in E$, such that

$$B \not\subset T_0', \qquad T_0' \not\subset B, \qquad B \cup T_0' \neq X.$$

B cannot contain x_0 ($B \cap T_0$ would give a contradiction on the definition of T_0), so B is as in Fig. 5.

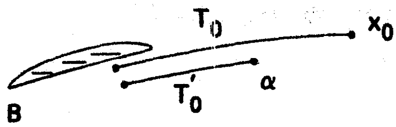

Fig. 5.

Then $|T_0 - B| \geq 2$ and we get clearly a contradiction. So we must suppose $|T_0| = 2$, and that is true independently from the choice of x_0.

Now let us consider a maximal chain $\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{p-1}, a_p\}$ of edges of cardinality 2 in H. Let us suppose $p \leq n-2$; then $\{a_1, \ldots, a_p\}$ is not vague: $\exists B \in E$, B contains a_p and doesn't contain a_1. $B' = B - \{a_1, \ldots, a_{p-1}\}$ is also in E, and we may assume $|B'| \neq 1$. But then B', nontrivial and containing a_p, must contain, because of the precedent reasoning, an edge B_0, also nontrivial and containing a_p, with cardinality 2. That means that the chain $(a_1, a_2, \ldots, a_p)$ is not maximal. Thus $p = n-1$ or n and the problem of unicity is achieved. □

3. A classification of the representations

We now describe a way of obtaining all representations of a circular hypergraph, generalizing the main part of our Theorem 2.1. Let r be a representation of a circular hypergraph $H = (X, E)$; r is a one-to-one function from X to the set $\{1, 2, \ldots, n = |X|\}$ in this order on a circle. For every rotation ρ (i.e., a circular permutation of $1, 2, \ldots, n$) $\rho \circ r$ is another representation of H, said equivalent to r. Let $\bar{R}(H)$ be the quotient of $R(H)$ by this equivalence. The image of r in $\bar{R}(H)$ is denoted by $\bar{r}$.

The inversions v and v' defined in Section 2 are particular cases of transformations that can be associated to vague subsets of H. Let $I = \{a, a+1, \ldots, b\}$ be a circular interval of $1, 2, \ldots, n$ (the integers here are modulo n). The *inversion* v_I is the permutation of $1, 2, \ldots, n$ defined by

$$v_I(k) = k \qquad \text{for } k \notin I,$$
$$v_I(k) = a + b - k \quad \text{for } k \in I.$$

When $r^{-1}(I)$ is a vague subset of H, $v_I \circ p$ is still a representation of H.

More generally, suppose A is a vague subset of H. $r(A)$ admits a unique decomposition $r(A) = I_1 \cup I_2 \cup \cdots \cup I_k$ into disjoint non-empty intervals such that $I_i \cup I_j$ is never an interval for $1 \leq i < j \leq k$. The inverse images $r^{-1}(I_i)$ form a partition of A which is called the *r-decomposition* of A. We define the transformation φ_A by

$$\varphi_A(r) = v_{I_1} \circ v_{I_2} \circ \cdots \circ v_{I_k} \circ r.$$

It is easy to verify that every $r^{-1}(I_i)$ is a vague subset of H; Thus $\varphi_A(r) \in R(H)$. A transformation φ_A preserves the equivalence of representations, hence induces a transformation Φ_A on $\bar{R}(H)$. The operators Φ_A (A: vague subset of H) generate a group of operators on $\bar{R}(H)$ which is denoted by $\Phi(H)$.

When the subsets of cardinality 0, 1, $n-1$ or n are the only vague subsets of H, one easily sees that $\Phi(H)$ is isomorphic to the group D_n introduced in Section 2. Thus the following result generalizes the 'if' part of Theorem 2.1.

Theorem 3.1. *$\Phi(H)$ acts transitively on $\bar{R}(H)$.*

Proof. What we have to prove is that r' and r'' being given in $R(H)$, it is possible to pass from $\bar{r}'$ to $\bar{r}''$ using a chain of operators $\Phi_{V_1} \circ \cdots \circ I_{V_k}$, *where* $V_1, \ldots, V_k$ are vague subsets of H. We proceed by induction on $|X|$ (straightforward verification until $|X| = 3$).

Lemma 3.2. *Let $r \in R(H)$ and A a vague subset of H. Then for some $\Phi \in \Phi(H)$, $\Phi(\bar{r})(A)$ (defined modulo rotations) is an interval.*

Proof. The assertion is trivial when $r(A)$ is an interval. If $r(A)$ is not an interval, let $A_1, A_2, \ldots, A_k$ the r-decomposition of A. $X - A$ is vague and admits a r-decomposition $B_1, \ldots, B_k$ such that $r(A_i \cup B_i)$ and $r(B_i \cup A_{i+1})$ are intervals for $i = 1 \cdots k$. For every i, $C_i = A_i \cup B_i$ is easily checked to be a vague subset of H. $\Phi_{C_i}(r)$ maps A in no more than $k-1$ disjoint intervals. An iteration of this process yields the required operator. □

Returning to the proof of Theorem 3.1, we consider two representations r' and r'' of H. $|X|$ is at least 4 and the theorem is supposed true for all interval hypergraphs with less than $|X|$ vertices.

In the case H does not admit a vague subset A with $|A| \notin \{0, 1, n-1, n\}$, the conclusion follows from Theorem 2.1. So we suppose that A is a non-trivial vague subset. By our Lemma 3.2, we may suppose that $r'(A)$ is an interval. Let us consider the two circular hypergraphs $H_1 = H(A)$ and $H_2 = H(X - A)$ obtained from H, as in the proof of Theorem 1.1, by fusing the vertices of A (respectively of $X - A$) into a single vertex a_1 (respectively a_2). Every representation r of H induces representations r_1 of H_1 and r_2 of H_2. Passing to the quotient, we see that the mapping $\bar{r} \to (\bar{r}_1, \bar{r}_2)$ is in fact one to one from $\bar{R}(H)$ onto $\bar{R}(H_1) \times \bar{R}(H_2)$.

For every subset Y of X_1, we define Y^1 by

$$Y^1 = Y \qquad \text{if } a_1 \notin Y,$$

$$Y^1 = (Y - \{a_1\} \cup (X - A) \qquad \text{if } a_1 \in Y.$$

It is easily seen that Y^1 is a vague subset of H if Y is a vague subset of H_1; $\bar{r}_1(Y)$ is an interval if and only if $\bar{r}(Y)$ is an interval. Thus, for every vague subset Y of H_1 such that $\bar{r}_1(Y)$ is an interval, we have

$$\Phi^1_Y(\bar{r}_1) = \Psi^1_Y(\bar{r}), \tag{3.1}$$

where Φ^1_Y is the operator of $\Phi(H_1)$ corresponding to Y and Ψ^1_Y is defined to be Φ_{Y^1} if $a_1 \notin Y$ and to be $\Phi_{X-A} \circ \Phi_{Y^1}$ if $a_1 \in Y$.

When Y is a vague subset of H_1 such that $\bar{r}_1(Y)$ is not an interval, we consider its r_1-decomposition $Y = Y_1, \ldots, Y_k$. The subsets Y_i are vague in H_1 and only one of them, say Y_1 may contain a_1. Hence (3.1) is still valid, with Ψ^1_Y defined by

$$\Psi^1_Y = \Psi^1_{Y_1} \circ \Phi_{Y^1_2} \circ \Phi_{Y^1_3} \circ \cdots \circ \Phi_{Y^1_k}.$$

Mutatis mutandis, the same assertions hold for H_2 and we have

$$\Phi^2_Y(\bar{r}_2) = \Psi^2_Y(\bar{r}) \tag{3.2}$$

Now, let us suppose that $\bar{r}'$ and $\bar{r}''$ are elements of $\bar{R}(H)$, inducing respectively $\bar{r}'_1$ and $\bar{r}''_2$ in $\bar{R}(H_1)$, $\bar{r}'_2$ and $\bar{r}''_2$ in $\bar{R}(H_2)$. Since $|A|>1$ and $|X-A|>1$, H_1 and H_2 have less vertices than H and the induction assumption can be applied: there are vague subsets $B_1, B_2, \ldots, B_p$ of H_1 and vague subsets $C_1, C_2, \ldots, C_q$ of H_2 such that

$$\bar{r}''_1 = \Phi^1_{B_1} \circ \Phi^1_{B_2} \circ \cdots \circ \Phi^1_{B_p}(\bar{r}'_1), \qquad \bar{r}''_2 = \Phi^2_{C_1} \circ \Phi^2_{C_2} \circ \cdots \circ \Phi^2_{C_q}(\bar{r}'_2).$$

Applying (3.1) and (3.2) to the B_i's and to the C_i's, we obtain

$$\bar{r}'' = \Psi^1_{B_1} \circ \Psi^1_{B_2} \circ \cdots \circ \Psi^1_{B_p} \circ \Psi^2_{C_1} \circ \Psi^2_{C_2} \circ \cdots \circ \Psi^2_{C_q}(\bar{r}').$$

This equality concludes the proof. □

3.1. *Application to the interval representation problem*

If $H=(X, E)$ is an interval-representable hypergraph, we say that $A \subset X$ is interval vague, if we have:

$$\forall B \in E, \quad B \cap A = B, A \text{ or } \emptyset.$$

An interval representation of H is a one-to-one function p from X to the ordered set $(1, 2, \ldots, n = |X|)$ such that

$$\forall A \in E, \quad p(A) \text{ is an interval.}$$

We denote by $I(H)$ the set of the interval representations of H.

We denote by H the smallest hypergraph (X, E') such that

$E \subset E'$ and $X \in E'$;
$E_1, E_2 \in E'$, $E_1 \cap E_2 \neq \emptyset \Rightarrow (E_1 \cup E_2 \in E'$ and $E_1 \cap E_2 \in E')$;
$E_1 \not\subset E_2 \Rightarrow E_2 - E_1 \in E'$.

An interval vague edge A of H generates a transformation Φ_A on $I(H)$:
If $p \in I(H)$; $p(A)=\{i_1, i_2, \dots, i_k\}$, then $p'=\Phi_A(p)$ is defined by

$$p'(x)=p(x) \quad \text{for } x \notin A,$$
$$p'(x)=i_{k+1-u} \quad \text{for } x \in A \text{ and } p(x)=i_u.$$

The set of all the transformations Φ_A (A interval vague edges of H) generates a group $S(H)$, and we can get from Theorem 1.1 and Theorem 3.1 the following corollary, which is in fact another way to state Duchet's result in [2]:

Corollary 3.3. *If H is an interval-representable hypergraph, $S(H)$ acts transitively on $I(H)$.*

As a consequence, we may remark that the number of interval representations of H will be reduced to 2 if and only if the only interval vague subsets in H are $\emptyset$, X and the subsets with cardinality 1.

4. Connection with the problem of characterizing the circular graphs

We recall that a graph $G=(X, E)$ is *triangulated* when every cycle of length ≥ 4 of G possesses a chord (for a survey on triangulated graph, see [4]).

G is *circular* [9, 10], when it is possible to find a function p from X to the set of the closed intervals of a topological circle such that

$$[x, y] \in E \Leftrightarrow p(x) \cap p(y) \neq 0.$$

p is a *circular representation* of G. No characterization of circular graphs is known.

Let $\mathscr{C}$ be the set of maximal (for the inclusion) cliques of a graph $G=(X, E)$. For $x \in X$, we set

$$U_x = \{c \in \mathscr{C} : x \in \mathscr{C}\}.$$

Fulkerson and Gross [5] proved the following equivalence: G is an interval graph if and only if $(X, (U_x)_{x \in X})$ is an interval-representable hypergraph (see [4] for some extensions). Our purpose here is to establish similar connections between circular-representable hypergraphs and circular graphs.

We call a K-system, a couple (G, E_0) where $G=(X, E)$ is a graph and E_0 is a set of G-edges. (G, E_0) will be said K-representable if we can find a function p from X to the set of the *closed* intervals of a circle K such that, for every x, $y \in X$,

$$[x, y] \in E \Leftrightarrow p(x) \cap p(y) \neq \emptyset, \qquad [x, y] \in E_0 \Leftrightarrow p(x) \cup p(y) = K.$$

Lemma 4.1. *Let p a K-representation of a K-system (G, E_0) and suppose $C \subset X$ is a clique of $G=(X, E)$. Then $\bigcap_{x \in C} p(x) \neq \emptyset$ if and only if we have both the* *properties*

(i) $p(x)\cap p(y)\cap p(z)\neq\emptyset$ *for all* x, y, $z\in C$.

(ii) *The graph* $(C, E-E_0)$ *is triangulated.*

Proof. In order to show that (ii) is a necessary condition, suppose we have $\bigcap_{x\in C}p(x)\neq\emptyset$ and that $(C, E-E_0)$ possesses a cycle $x_1\cdots x_kx_1$ $(k\geq 4)$ with no chords. Then, setting $I_i=p(x_i)$, we have a family $I_1,\ldots,I_k$ of circular closed intervals of the circle K such that

$$\bigcap_{i=1}^{k} I_i\neq\emptyset,\qquad I_i\cup I_{i+1}\neq K,$$

$$I_2\cup I_k=K\quad\text{and}\quad I_1\cup I_j=K\quad\text{for } 3\leq j\leq k-1.$$

This is impossible (the verification is straightforward and left to the reader).

Conversely, suppose that (i) and (ii) are satisfied by a K-system (G, E_0), and a clique C of G. We prove $\bigcap_{x\in C}p(x)\neq\emptyset$ by induction on $|C|$. By a well-known theorem due to Dirac (see [4]), every triangulated graph possesses a *simplicial vertex*, i.e., a vertex whose neighbours are pairwise adjacent. Let x_0 a simplicial vertex of $(C, E-E_0)$. By the induction hypothesis, the intervals $p(x)$, for $x\in C-x_0$, have a common point λ. If $\lambda\in p(x_0)$, we have finished. If $\lambda\notin p(x_0)$, let μ and ν the ends of $p(x_0)$. Put

$$M=\{x\in C-x_0:\mu\in p(x)\},\qquad N=\{x\in C-x_0:\nu\in p(x)\}.$$

Our hypotheses imply $M\cup N=C-x_0$. If $m\in M-N$ and $n\in N-M$, we have $[x_0, m]\in E-E_0$ and $[x_0, n]\in E-E_0$. Since x_0 is a simplicial vertex of $(C, E-E_0)$, we have $[m, n]\in E-E_0$; thus $p(m)\cup p(n)\neq K$. But $\lambda\in(p(m)\cap p(n))-p(x_0)$. Hence $p(m)\cap p(n)\cap p(x_0)=\emptyset$, in contradiction with (i). We conclude that either $M\subset N$ or $N\subset M$. Thus $\bigcap_{x\in C}p(x)$ contains μ or ν. $\square$

Given a graph $G=(X, E)$, we say that a triangle T in G (eventually degenerate) is *attractive* if, every adjacent or equal vertices x and y in $X-T$ have a common neighbour in T. If T is in fact an edge ($|T|=2$), we say that T is an *attractive edge.*

Lemma 4.2. *Let* G *a circular graph and* E_0 *the set of its attractive edges. Then the* K*-system* (G, E_0) *admits a* K*-representation.*

Proof. Let $[a, b]\in E_0$. It suffices to prove that a representation r of G such that $r(a)\cup r(b)\neq K$ can be transformed into a representation p of G such that $p(a)\cup p(b)=K$; the lemma will follow by induction on $|E_0|$. Define the sets

$$A=\{x\in X: x \text{ is adjacent to } a \text{ and not to } b\},$$

$$B=\{x\in X: x \text{ is adjacent to } b \text{ and not to } a\}.$$

Since $[a, b]$ is attractive, no edge joins A to B. Hence, the set $r(a) \cup \bigcup_{x \in A} r(x) \cup r(b) \cup \bigcup_{x \in B} r(x) = L$ does not cover K. $K - L$ is an interval (α, β) where α is in some $r(x)$ for $x \in A$ or in $r(a)$ and β is in some $r(y)$ for $y \in B$ or in $r(b)$.

Let $(\alpha, \gamma]$ and $[\gamma, \beta)$ denote respectively the subintervals of (α, β) determined by a point γ chosen in (α, β). We define p by

$$p(x) = r(x) \quad \text{for } x \notin \{a, b\},$$

$$p(a) = r(a) \cup \bigcup_{x \in A} r(x) \cup (\alpha, \gamma], \qquad p(b) = r(b) \cup \bigcup_{x \in B} r(x) \cup [\gamma, \beta).$$

The reader will easily verify that p is a circular representation of G and that $p(a) \cup p(b) = K$. □

Let E_0 be the set of attractive edges of z graph $G = (X, E)$; we define the hypergraph $H(G)$ in the following manner: the vertices of $H(G)$ are the cliques C of G that are maximal (for the inclusion) with the property (ii); the set of these cliques is denoted by $\mathcal{A}(G)$. The edges of $H(G)$ are the subsets of $\mathcal{A}(G)$ of the form $A_x = \{C \in \mathcal{A}(G) : x \in C\}$.

Theorem 4.3. *If $G = (X, E)$ is a finite graph such that every attractive triangle of G contains an attractive edge of G, then the following equivalence is true: G is a circular graph if and only if $H(G)$ is a circular representable hypergraph.*

Proof. The 'if' part is trivial since G coincides with the line-graph of $H(G)$. In order to prove the 'only if' part, we suppose that G is circular representable and we call E_0 the set of attractive edges of G. By Lemma 4.2, (G, E_0) admits a K-representation p. Since every triangle of G contains an attractive edge, we have $p(x) \cap p(y) \cap p(z) \neq \emptyset$ for every triangle $\{x, y, z\}$. By Lemma 4.1, for every $C \in \mathcal{A}(G)$, the intervals $p(x)$ $(x \in C)$ have a common point C. The hypergraph induced on $\{\lambda_C : C \in \mathcal{A}(G)\}$ by the intervals $p(x)$ is isomorphic to $H(G)$. □

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